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Relations between univalent and orthogonal polynomials ☆

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Abstract

Two sequences of polynomials are studied. One satisfies a three term recurrence relation for specific parameters and another a para-orthogonality property. Using the fact that these polynomials have their zeros lying on the unit circle and some other properties, we establish a criterion in order that the polynomials be univalent in the open unit disk.

Keywords: Univalent polynomials; Para-orthogonal polynomials; Szegő polynomials; Zeros

1. Introduction

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, $\mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\}$ the unit circle and $\mathcal{A}(\mathbb{D})$ the set of all analytic functions in $\mathbb{D}$. A function f is *univalent* in $\mathbb{D}$ if it is one-to-one in $\mathbb{D}$. We consider several sets of functions defined in $\mathbb{D}$: $\mathcal{N}_n(\mathbb{D})$ denotes the set of all nth degree polynomials having the form

$$p_n(z) = z + \sum_{k=2}^{n} b_k z^k, \tag{1.1}$$

while $\mathcal{S}_n(\mathbb{D})$ denotes the subset of $\mathcal{N}_n(\mathbb{D})$ formed by the univalent polynomials.

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In addition, $\mathcal{K}_n(\mathbb{D})$ will denote the subset of $\mathcal{N}_n(\mathbb{D})$ composed of those elements whose image is convex. Finally,

$$\mathcal{C}_n(\mathbb{D}) = \{p \in \mathcal{N}_n(\mathbb{D})\text{: } \operatorname{Re}(p'/e^{i\alpha}\varphi') \geqslant 0 \text{ for some } \varphi \in \mathcal{K}_n(\mathbb{D}) \text{ and } \alpha \in \mathbb{R}\}.$$

Elements of $\mathcal{K}_n(\mathbb{D})$ are called *convex* univalent polynomials while those of $\mathcal{C}_n(\mathbb{D})$ are called *close-to-convex* univalent polynomials.

Our purpose in this work is to investigate the existence of some relations between univalent polynomials and orthogonal polynomials on the unit circle. The remainder of the section contains the basic material needed in the forthcoming sections.

A powerful tool in the study of orthogonal polynomials is the three term recurrence relation. Let $\{Q_m\}_{n=1}^{\infty}$ be a sequence of polynomials generated by the relation

$$Q_{m+1}(z) = (z + \beta_{m+1})Q_m(z) - \alpha_{m+1} z Q_{m-1}(z), \quad m \geqslant 1, \tag{1.2}$$

with $Q_0 = 1$ and $Q_1(z) = z + \beta_1$, where α_m and β_m are complex numbers satisfying $\alpha_{m+1} \neq 0 \neq \beta_m$, $m \geqslant 1$.

In Section 2, we will mention some results about the zeros of polynomials that satisfy a relation of the form above.

It is known that for a particular choice of α_{m+1} and β_{m+1}, $m \geqslant 1$, the corresponding polynomials satisfy a orthogonality property. For example we mention the Szegő polynomials $\{\rho_n\}_{n=0}^{\infty}$ introduced in [9].

In 1989, Jones et al. [5] studied polynomials related with Szegő polynomials $\{\rho_n\}_{n=0}^{\infty}$, the so-called *para-orthogonal* polynomials

$$B_n(z, \omega_n) = \rho_n(z) + \omega_n \rho_n^*(z), \quad |\omega_n| = 1, \tag{1.3}$$

where $\rho_n^*(z) = z^n \overline{\rho}_n(1/z)$ are the reciprocal polynomials. These polynomials are associated with a measure ν and a parameter ω_n. In Sections 4 and 5 we will discuss many properties of Szegő and para-orthogonal polynomials.

One very interesting property of these polynomials is that B_n has n simple zeros all lying on $\mathbb{T}$ [5]. In relation (1.2) for particular cases of β_{m+1} and α_{m+1} the polynomials Q_m have the same property. This behaviour of the zeros of these polynomials and Suffridge's result [6,8] about the critical points of univalent polynomials motivated us to search for a connection between these two objects. Suffridge's result is given in Section 3.

In summary, we investigate univalence criteria for polynomials P_m whose derivatives P_m' satisfy a relation of the form (1.2). In the sequel, we consider polynomials U_n whose derivatives U_n' are polynomials like in (1.3).

Our main contributions in this work are given by Theorems 1–3, where we establish a very simple and nice univalence criteria for the polynomials considered. These results are useful because the conditions are imposed only on parameters of the three term recurrence relation or reflection coefficients. They are given in Sections 3 and 4. Finally, in Section 5, we give some examples that fit into our results.

2. Bounds for zeros of polynomials satisfying a recurrence relation

In this section, we enunciate some results about polynomials that satisfy a three term recurrence relation. The zeros of these polynomials belong to bounded regions that depend on the coefficients of the recurrence relation.

To begin, we quote a theorem that gives some information about the zeros of polynomials Q_m, when all the α_m are positive and all the β_m are equal to some positive constant. A proof in the case $\beta_m = 1$ can be found in [1].

First, we give a concept used in this result, a sequence $\{c_n\}_{n=1}^{\infty}$ is said to be a chain sequence, if there exists a second sequence $\{g_k\}_{k=0}^{\infty}$ such that $0 \leqslant g_0 < 1$, $0 \leqslant g_n < 1$, $n \geqslant 1$, and $c_n = (1 - g_{n-1})g_n$, $n \in \mathbb{N}$.

Theorem A. [3, Theorem 3.3] *Let $\beta_k = \beta > 0$ and $\alpha_{k+1} > 0$ for $k = 1, 2, 3, \dots, m$. Then, the zeros of any Q_k, $1 \leqslant k \leqslant m$, are distinct (except for a possible double zero at $z = \beta$) and lie on $\mathcal{T}(\beta) \cup (0, \infty)$, where*

$$\mathcal{T}(\beta) = \{z\colon\ z = \beta e^{i\theta},\ 0 < \theta < 2\pi\}.$$

In particular, if $\{\alpha_{k+1} 4^{-1} \beta^{-1}\}_{k=1}^{m-1}$ is a positive chain sequence then all the zeros are distinct and lie on the open circle $\mathcal{T}(\beta)$.

We collect some particular cases in the following corollary.

Corollary B. [3, Corollary 3.3.1] *Let $\beta_k = \beta > 0$, $1 \leqslant k \leqslant m$. Then the following hold:*

(i) *if $0 < \alpha_{k+1} \leqslant \beta$, $1 \leqslant k \leqslant m - 1$, then all the zeros of Q_k, $1 \leqslant k \leqslant m$, are on $\mathcal{T}(\beta)$;*
(ii) *in particular, for $\varepsilon > 0$, if $0 < \alpha_{k+1} \leqslant \beta - \varepsilon$, $1 \leqslant k \leqslant m - 1$, then the zeros of any Q_k, $1 \leqslant k \leqslant m$, are on the arc of $\mathcal{T}(\beta)$ that stays outside the parabolic region*

$$\mathcal{P}^+(\varepsilon) \equiv \{z = x + iy \in \mathbb{C}\colon\ y^2 \leqslant 4\varepsilon(\varepsilon + x),\ x > -\varepsilon\};$$

(iii) *again, if $0 < \alpha_{k+1} \leqslant \beta$, $1 \leqslant k \leqslant m - 1$, let $\kappa_m = \max_{1 \leqslant k \leqslant m-1} \sqrt{\alpha_{k+1}/\beta}$. Then all the zeros of Q_k, $1 \leqslant k \leqslant m$, are on the open arc*

$$\mathcal{T}(\beta, \theta_m) = \{z\colon\ z = \beta e^{i\theta},\ \theta_m < \theta < 2\pi - \theta_m\}$$

where $\theta_m = 2\arccos(\kappa_m \cos(\pi(m+1)^{-1}))$;
(iv) *for $\varepsilon > 0$ suppose that $0 < \alpha_{k+1} \leqslant \beta + \varepsilon$, $1 \leqslant k \leqslant m - 1$. If Q_m has any zeros outside $\mathcal{T}(\beta)$, then these zeros lie inside the interval $(\beta^2/b, b)$, where $b = (\sqrt{\beta + \varepsilon} + \sqrt{\varepsilon})^2$.*

The last theorem and corollary were proved in [3].

3. Critical points of close-to-convex polynomials

The next result is one of Suffridge's remarkable results on polynomials with zeros on $\mathbb{T}$ and separated by a minimum angle. The proof is given in [8] and some comments about it can be found in [6, p. 243].

Theorem C. [8, Theorem 2] *Let P be a polynomial of degree n with all its critical points on $\mathbb{T}$. If each pair of critical points is separated by an angle of at least $2\pi/(n+1)$, then P is close-to-convex and hence univalent in $\mathbb{D}$. Conversely, if P is close-to-convex in $\mathbb{D}$, then the critical points have a separation angle of at least $2\pi/(n+1)$.*

To obtain our first result we use the properties about polynomials that satisfy a relation like in (1.2) and Corollary B(iii) along with Theorem C. The idea is based on the fact that while

Corollary B restricts the area where the zeros are located, each pair of critical points of close-to-convex polynomials must have a minimal separation angle. Here, we consider polynomials P_m whose derivatives satisfy

$$P'_m = Q_{m-1}, \tag{3.1}$$

where Q_{m-1} are polynomials of the form (1.2).

Theorem 1. *Let P_{m+1} be a polynomial and assume that P'_{m+1} satisfies the recurrence relation* (1.2) *with $\beta = 1$ and $0 < \alpha_{k+1} \leqslant 1$, $1 \leqslant k \leqslant m-1$. In addition, assume that*

$$(\pi - \theta_m) \leqslant \frac{\pi(m-1)}{m+2}, \tag{3.2}$$

in which $\theta_m = 2\arccos(\kappa_n \cos(\pi(m+1)^{-1}))$ and $\kappa_m = \max_{1\leqslant k\leqslant m-1} \sqrt{\alpha_{k+1}}$. Then P_{m+1} is not close-to-convex in $\mathbb{D}$.

Proof. Using Corollary B(iii) we have that the arc

$$\mathcal{T}(1, \theta_m) = \{z\colon\ z = e^{i\theta},\ \theta_m < \theta < 2\pi - \theta_m\},$$

where $\theta_m = 2\arccos(\kappa_m \cos(\pi(m+1)^{-1}))$ and $\kappa_n = \max_{1\leqslant k\leqslant m-1} \sqrt{\alpha_{k+1}}$, contains all the zeros of P'_{m+1} and the quantity $2(\pi - \theta_m)$ estimates the arc length that contain all these zeros. But, the assumption given by inequality (3.2) implies that

$$2(\pi - \theta_m) \leqslant (m-1)\frac{2\pi}{m+2}.$$

This mean that the separation angle of at least one pair of zeros of P'_{m+1} must be smaller than $2\pi/(m+2)$. Theorem C closes the proof. □

4. Properties about the zeros of para-orthogonal polynomials

In this section we list some results about the angular distance between consecutive zeros of para-orthogonal polynomials.

First, we define the Szegő polynomials formally. Let ν be a positive measure on the unit circle. This means that $\nu(e^{i\theta})$, defined on $0 \leqslant \theta \leqslant 2\pi$, is a real, bounded and non-decreasing function with infinitely many points of increase and such that all the moments

$$\mu_m = \int z^m \, d\nu(z) = \int_0^{2\pi} e^{im\theta} \, d\nu(e^{i\theta}), \quad m \geqslant 0,$$

exist. We consider Szegő polynomials $\{\rho_n\}_{n=0}^{\infty}$ associated with the measure ν defined by

$$\int \rho_n(z)\overline{\rho_m(z)}\, d\nu(z) = 0, \quad n \neq m.$$

Basic information about them is to be found in [10].

The Szegő polynomials are known to satisfy the following recurrence relations, given here in term of monic polynomials,

$$\rho_{n+1}(z) = z\rho_n(z) + a_{n+1}\rho_n^*(z), \quad n \geqslant 0, \tag{4.1}$$

and

$$\big(1-|a_{n+1}|^2\big)z\rho_n(z)=\rho_{n+1}(z)-a_{n+1}\rho_{n+1}^*(z),\quad n\geqslant 0,\tag{4.2}$$

where $\rho_n^*(z)=z^n\overline{\rho}_n(1/z)$ are the reciprocal polynomials. The numbers $a_n=\rho_n(0)$ satisfy $|a_n|<1$, $n\geqslant 1$, and are known as the Szegő reflection coefficients. Furthermore, the zeros of the Szegő polynomials are all inside $\mathbb{D}$.

If the reflection coefficients are non-zero, then the Szegő polynomials satisfy

$$\rho_{n+1}(z)=\left(z+\frac{a_{n+1}}{a_n}\right)\rho_n(z)-\frac{a_{n+1}}{a_n}\big(1-|a_n|^2\big)z\rho_{n-1}(z),\quad n\geqslant 1,\tag{4.3}$$

where $\rho_1(z)=(z+a_1a_0^{-1})$, with $a_0=1$. Hence these Szegő polynomials satisfy a three term recurrence relation of the form (1.2) with $\beta_n a_n/a_{n-1}$ and $\alpha_{n+1}=\frac{a_{n+1}}{a_n}(1-|a_n|^2)$, $n\geqslant 1$.

Considering para-orthogonal polynomials defined in (1.3) the word "para" is used to indicate that, for $n\geqslant 2$, the orthogonality property

$$\int_{\mathbb{T}}B_n(z,\omega_n)\overline{z^k}\,d\nu(z)=\int_0^{2\pi}B_n\big(e^{i\theta},\omega_n\big)e^{-ik\theta}\,d\nu\big(e^{i\theta}\big)=0,\quad 1\leqslant k\leqslant n-1,$$

is weak (unlike the Szegő polynomial ρ_n, B_n lacks the property for $k=0$).

Let $\zeta_{k,n}$, $1\leqslant k\leqslant n$, be the zeros of the para-orthogonal polynomial B_n of degree n. The following theorem is a particular case of Theorem 6 in [2]. This result is based on relation (1.3) and it imposes conditions only on the reflection coefficients a_n of the Szegő polynomials. In this section we consider that $|a_n|<1$ and non-zero for all n.

Theorem D. [2] *Let the Szegő reflection coefficients a_n be such that $|a_n|\leqslant r<1$ for $n>N$. Setting $s=(1-r)/(1+r)$ and $\zeta_{n+1,n}=\zeta_{1,n}$, then*

$$2\pi s^{n-N}\frac{1-s}{1-s^{n-N+1}}\leqslant\arg(\zeta_{k+1,n})-\arg(\zeta_{k,n})\leqslant 2\pi\frac{1-s}{1-s^{n-N+1}}.\tag{4.4}$$

In the sequel the reflection coefficients obey the condition

$$\sup_{n\geqslant 1}a_n=r\quad\text{and}\quad r<1.\tag{4.5}$$

The following result gives a condition that guarantees that an element of the sequence $\{U_n\}_{n=1}^{\infty}$ is close-to-convex in $\mathbb{D}$. In general, this condition is satisfied for only first elements of this sequence. The polynomials U_n considered here satisfy the condition

$$U_{n+1}'(z)=B_n(z,\omega_n).\tag{4.6}$$

Theorem 2. *Let U_{k+1} be a polynomial, whose derivative is a para-orthogonal polynomial. Setting $s=(1-r)/(1+r)$ and if*

$$s^k-\left(\frac{k+2}{k+1}\right)s^{k-1}+\frac{1}{k+1}\leqslant 0,\tag{4.7}$$

then U_{k+1} is close-to-convex.

Proof. From (4.5) the inequalities (4.4) hold for all n, $n > 1$. The condition (4.7) is equivalent to

$$2\pi s^{k-1}\frac{1-s}{1-s^k} \geqslant \frac{2\pi}{k+2}.$$

Therefore, putting $n = k$ and $N = 1$ in (4.4) and setting $\zeta_{1,k} = \zeta_{k+1,k}$, we conclude that

$$\arg(\zeta_{i+1,k}) - \arg(\zeta_{i,k}) \geqslant \frac{2\pi}{k+2}, \quad i = 1, 2, \ldots, k.$$

Hence, each pair of zeros of the derivative of P_{k+1} is separated by an arc length of γ satisfying $\gamma \geqslant 2\pi/(k+2)$. From Theorem C the statement follows. □

It is worth pointing out that this result is useful for cases when the reflection coefficients satisfy (4.5) and r is a value close to zero.

We restrict the last result for particular cases of reflection coefficients a_n and we are able to extend the univalence criteria for all elements of the sequence $\{U_n\}_{n=1}^{\infty}$.

Theorem 3. *Let U_{n+1} be polynomials, whose derivatives are para-orthogonal polynomials. Suppose that the inequalities*

$$0 < |a_n| \leqslant \frac{2}{n^2+n-2}, \quad n > 1, \tag{4.8}$$

are valid. Then U_{n+1} is close-to-convex.

Proof. By hypotheses (4.8), we obtain for $s = (1-r)/(1+r)$ that

$$\frac{n^2+n-4}{n(n+1)} \leqslant s < 1.$$

Simplifying, we denote the function

$$f_n(s) = s^n - \left(\frac{n+2}{n+1}\right)s^{n-1} + \frac{1}{n+1}, \quad n > 1. \tag{4.9}$$

This polynomial has two positive roots, one at $\zeta_1 = 1$ and the other zero ζ_2 satisfies

$$\zeta_2 \leqslant \frac{n^2+n-4}{n(n+1)}.$$

Since f_n attains the minimum value in $\varepsilon = (n^2+n-2)/n(n+1)$ and $f_n(\varepsilon)$ is negative, we can find the rate in the last inequality by the distance $(2\varepsilon - \zeta_1)$. On that occasion, we can verify that the function f_n remains negative in $(n^2+n-4)/n(n+1)$. In fact, using that $(1+x)^n \geqslant 1+nx$ for $x > -1$ we obtain that

$$\begin{aligned} f_n\left(\frac{n^2+n-4}{n(n+1)}\right) &= \frac{1}{n+1} - \frac{(n+4)}{n(n+1)}\left(1 - \frac{4}{n(n+1)}\right)^n \\ &\leqslant -\frac{4}{n(n+1)}. \end{aligned}$$

Thus, from Theorem 2 the condition (4.7) is satisfied. Repeating this reasoning for all n, $n > 1$, the result follows. □

Note that the polynomial given in (4.9) satisfy a very interesting property, the relation

$$f_{n+1}(z)=\left(1+\frac{(n+1)}{(n+2)}\right)f_n(z)-\frac{n}{(n+1)}f_{n-1}(z),\quad n\geqslant 1,$$

where $f_0=15/3$ and $f_1(z)=z+1/2$. This three term recurrence relation is like in (1.2) with $\beta_{n+1}=(n+1)/(n+2)$ and $\alpha_{n+1}=n/(n+1)$. Probably, these polynomials possess other interesting properties that can be investigated.

5. Special cases of para-orthogonal polynomials

Para-orthogonal polynomial have one very useful property, namely *invariance*.

For $k\in\mathbb{C}$, $k\neq 0$, a polynomial q is called k-invariant if

$$q^*(z)=kq(z),\quad z\in\mathbb{C},$$

where $q^*(z):=z^n\overline{q}(1/z)$. A sequence of polynomials $\{q_n\}$ is said to be k_n-invariant if q_n is k_n-invariant for each n.

Since B_n, given in (1.3) is k_n-invariant, if β is a zero of B_n, then $1/\overline{\beta}$ is a zero of B_n whenever β is so. In particular, when $\omega_n=\pm 1$ the polynomials B_n are ± 1-invariant. This implies that the coefficients of B_n obey the following relation

$$(k+1)a_{k+1}=(n-k)\bar{a}_{n-k},\quad 0\leqslant k\leqslant n. \tag{5.1}$$

Since the Szegő polynomials are monic then the polynomials U_n given in (4.6) are monic too.

Considering in relation (5.1) that the all coefficients are real, it is easily seen that

$$e^{-in\theta/2}B_n\big(e^{i\theta},\omega_n\big),\quad 0\leqslant\theta\leqslant 2\pi,$$

are real trigonometric polynomial for $\omega_n=1$ and imaginary trigonometric polynomial for $\omega_n=-1$. Differentiating, we deduce that

$$\operatorname{Re}\left(e^{i\theta}\frac{B_n'(e^{i\theta},\pm 1)}{B_n(e^{i\theta},\pm 1)}\right)=\frac{n}{2},\quad \theta\in\mathbb{R}, \tag{5.2}$$

as long as $B_n(e^{i\theta},\pm 1)\neq 0$.

From (5.2) we obtain the following property of the map U_n given in (4.6), when we use $\omega_n=\pm 1$.

Lemma 1. *Let U_n be a polynomial as in* (4.6). *Then*

$$\operatorname{Re}\left(1+e^{i\theta}\frac{U_n''(e^{i\theta})}{U_n'(e^{i\theta})}\right)=\frac{n+1}{2}, \tag{5.3}$$

as long as $U_n'(e^{i\theta})\neq 0$.

Proof. It is sufficient to observe that

$$\operatorname{Re}\left(1+e^{i\theta}\frac{U_n''(e^{i\theta})}{U_n'(e^{i\theta})}\right)=\operatorname{Re}\left(1+e^{i\theta}\frac{B_{n-1}'(e^{i\theta},\pm 1)}{B_{n-1}(e^{i\theta},\pm 1)}\right),$$

and the formula (5.2) accomplishes the proof. □

This lemma gives a very interesting property about the map of U_n. It implies that the tangent line to the curve $\theta \in [-\pi, \pi] \to U_n(e^{i\theta})$, turns at a constant rate in the counterclockwise direction when θ runs from $-\pi$ to π, except at the cusps (the zeros of $B_{n-1}(e^{i\theta}, \pm)$) where it reverses direction. These cusps account for all the zeros of $U_n'(e^{i\theta})$. Another class of polynomials that satisfies the same property is given by Suffridge in [7].

Now we consider two special cases of para-orthogonal polynomials, namely

$$R_n(z,1) = \frac{\rho_n(z) + \rho_n^*(z)}{1 + a_n} \tag{5.4}$$

and

$$(z-1)R_n(z,-1) = \frac{\rho_{n+1}(z) - \rho_{n+1}^*(z)}{1 - a_{n+1}}, \tag{5.5}$$

for $n \geqslant 0$. The denominators are chosen in order to make the polynomials monic.

Clearly, we have that

$$2\rho_n(z) = (1 + a_n)R_n(z,1) + (1 - a_n)(z-1)R_{n-1}(z,-1), \quad n \geqslant 1.$$

Considering $|a_n| < 1$ in (4.1) and (4.2), the last formula leads to

$$2z\rho_{n-1}(z) = R_n(z,1) + (z-1)R_{n-1}(z,-1), \quad n \geqslant 1. \tag{5.6}$$

Since $|a_n| < 1$, the monic polynomials $R_n(z,\pm 1)$ satisfy $R_0(z,\pm 1) = 1$, $R_1(z,\pm 1) = z + 1$ and

$$R_{n+1}(z,\pm 1) = (z+1)R_n(z,\pm 1) - \alpha_{n+1,\pm 1}R_{n-1}(z,\pm 1), \quad n \geqslant 1, \tag{5.7}$$

with

$$\alpha_{n+1,1} = (1 + a_{n-1})(1 - a_n) > 0, \quad n \geqslant 1,$$

and

$$\alpha_{n+1,-1} = (1 - a_n)(1 + a_{n+1}) > 0, \quad n \geqslant 1.$$

This last result should be attributed to Delsarte and Genin [4]. Note that the reflection coefficients a_n are real if and only if the measure ν satisfies the symmetry $d\nu(1/z) = -d\nu(z)$. Hence, if $|a_n| < 1$, the para-orthogonal polynomials $R_n(z,\pm 1)$ satisfy the recurrence relation of the form (1.2) with $\beta_n = 1$ and $\alpha_{n+1} = \alpha_{n+1,\pm 1}$, $n \geqslant 1$.

As we mentioned before, all the zeros of para-orthogonal polynomials are distinct and lie on the unit circle. We verify this fact for two sequence of para-orthogonal polynomials described in (5.4) and (5.5). Since $|a_n| < 1$, the corresponding coefficients α_{n+1} given in (1.2) generate sequences $\{4^{-1}\alpha_{n+1,\pm 1}\}$ which are positive chain sequences with parameter sequences $\{g_{n,1} = 2^{-1}(1 - a_n)\}$ and $\{g_{n,-1} = 2^{-1}(1 - a_{n+1})\}$, respectively. That is,

$$(1 - g_{n,1})g_{n,1} = \frac{1}{4}\alpha_{n+1,1}, \quad n \geqslant 1, \tag{5.8}$$

and

$$(1 - g_{n-1,-1})g_{n,-1} = \frac{1}{4}\alpha_{n+1,-1}, \quad n \geqslant 1, \tag{5.9}$$

with $g_{0,1} = 0$ and $g_{0,-1} = 1 - \alpha_{2,1} = (1 + a_1)/2 < 1$. Therefore, Theorem A implies that the zeros of $R_n(z,\pm 1)$ are distinct and lie on the open unit circle $\mathcal{T}(1) = \{z\colon\ z = e^{i\theta},\ 0 < \theta < 2\pi\}$.

Example 1. We consider the Szegő polynomials with $a_n = q^n$, $n \geqslant 1$, where $0 \leqslant q < 1$. In the case $q > 0$, these polynomials satisfy $\rho_1(z) = z + q$ and the three term recurrence relation

$$\rho_{n+1}(z) = (z+q)\rho_n(z) - q\big(1-q^{2n}\big)z\rho_{n-1}(z), \quad n \geqslant 1. \tag{5.10}$$

Substituting a_n in the recurrence formula (5.7), we reach $R_1(z, \pm 1) = z + 1$ and the recurrence formulas

$$R_{n+1}(z,1) = (z+1)R_n(z,1) - \big(1+q^{n-1}\big)\big(1-q^n\big)R_{n-1}(z,1), \quad n \geqslant 1, \tag{5.11}$$

and

$$R_{n+1}(z,-1) = (z+1)R_n(z,-1) - \big(1-q^n\big)\big(1+q^{n+1}\big)R_{n-1}(z,-1), \quad n \geqslant 1. \tag{5.12}$$

Thus we have that

$$\int_{\mathbb{T}} \rho_n(z)\overline{\rho_m(z)}\, d\nu(z) = 0, \quad n \neq m,$$

in which $d\nu(z) = \pi^{-1}(q^2;q^2)_\infty(qz;q^2)_\infty(qz^{-1};q^2)_\infty(2iz)^{-1}\,dz$. The symbol $(a;q)_\infty$ stands for the infinite product $\prod_{k=0}^{\infty}(1-aq^k)$.

Note that $\{4^{-1}(1+q^{n-1})(1-q^n)\}_{n=1}^{\infty}$ and $\{4^{-1}(1-q^n)(1+q^{n+1})\}_{n=1}^{\infty}$ are positive chain sequences with parameter sequences given by $\{2^{-1}(1-q^n)\}_{n=0}^{\infty}$ and $\{2^{-1}(1+q^{n+1})\}_{n=0}^{\infty}$, respectively. Consequently, from (5.12), all the zeros of $R_n(z,\pm 1)$ lie on the open unit circle $\mathcal{T}(1)$.

In the case $R_n(z,-1)$, $1 \leqslant n \leqslant m$, Corollary B(iii) is applicable. We have

$$\kappa_m = \max_{1 \leqslant n \leqslant m-1} \sqrt{\alpha_{n+1}} = \big(1-q^m\big)\big(1+q^{m+1}\big)$$

and we conclude that all the zeros of $R_n(z,-1)$, $1 \leqslant n \leqslant m$, lie on the open unit circle

$$\mathcal{T}(\beta,\theta_m) = \big\{z\colon\ z = \beta e^{i\theta},\ \theta_m < \theta < 2\pi - \theta_m\big\},$$

where $\theta_m = 2\arccos(\kappa_m \cos(\pi(m+1)^{-1}))$.

Case 1: For $q = 0.9$, the region where the zeros of $R_{15}(z,-1)$ are located is illustrated in Fig. 1. In this case we also conclude that the polynomial P_{16} that satisfies the relation (3.1) is not close-to-convex in $\mathbb{D}$.

The constant in Theorem 1 can be estimated in the form

$$2(\pi - \theta_{15}) = 5.03286 < \frac{28\pi}{17}.$$

Furthermore, by Fig. 2 we can observe that P_{16} is not univalent in $\mathbb{D}$.

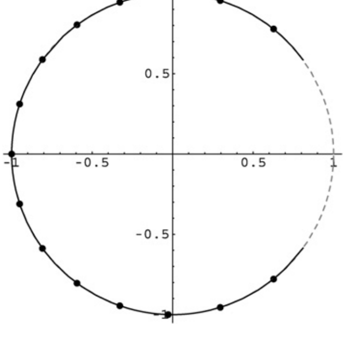

Fig. 1. Image of the arc that contain all zeros of $R_{15}(z,-1)$.

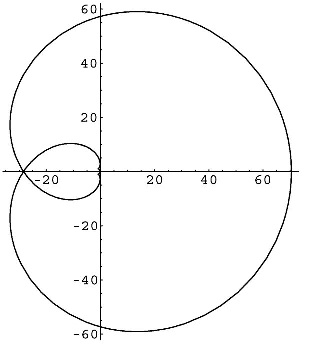

Fig. 2. Image of $\mathbb{T}$ by P_{16}.

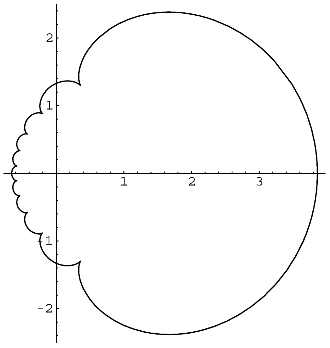

Fig. 3. Image of $\mathbb{T}$ by U_{11}.

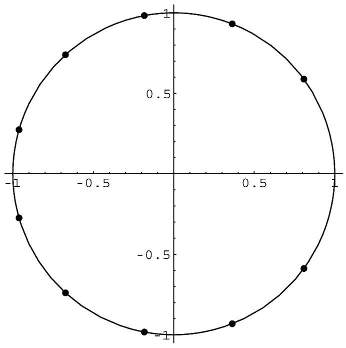

Fig. 4. Image of the distribution of all zeros of $R_{10}(z, -1)$.

Case 2: Setting $q = 0.5$, from Theorem 3 we obtain that the inequalities (4.8) are satisfied for $n > 1$. Therefore, the polynomials U_{n+1} given by the relation (4.6) are univalent in $\mathbb{D}$. This result is confirmed for $n = 10$ in Fig. 3. In Fig. 4 we can observe the distribution of the zeros of $R_{10}(z, -1)$.

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